1089. Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Prove that if $x,y,z \geq 1$, then

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \geq \frac{1}{1+\sqrt{xy}}+\frac{1}{1+\sqrt{yz}}+\frac{1}{1+\sqrt{zx}} \geq \frac{3}{1+\sqrt[3]{xyz}}.$$

Solution by Arkady Alt , San Jose ,California, USA.

First we will prove inequality

(**1**) $\quad \dfrac{1}{1+x}+\dfrac{1}{1+y} \geq \dfrac{2}{1+\sqrt{xy}}, x,y \geq 1.$

Let $p := x+y, q := xy$. Then $p^2 \geq 4q$,$p \geq 2, q \geq 1$, $\dfrac{1}{1+x}+\dfrac{1}{1+y} = \dfrac{2+x+y}{1+x+y+xy} = \dfrac{2+p}{1+p+q} = 1-\dfrac{q-1}{1+p+q}$.

Since $\dfrac{2+p}{1+p+q}$ increasing in $p>0$ ($\dfrac{2+p}{1+p+q} = 1-\dfrac{q-1}{1+p+q}$ and $q \geq 1$) and $p \geq 2\sqrt{q}$ then $\dfrac{2+p}{1+p+q} \geq \dfrac{2+2\sqrt{q}}{1+2\sqrt{q}+q} = \dfrac{2}{1+\sqrt{q}}$.

Using inequality (**1**) we obtain

$$\sum_{cyc}\left(\frac{1}{1+x}+\frac{1}{1+y}\right) \geq \sum_{cyc}\frac{2}{1+\sqrt{xy}} \Leftrightarrow \sum_{cyc}\frac{1}{1+x} \geq \sum_{cyc}\frac{1}{1+\sqrt{xy}}.$$

Let $u := \sqrt[3]{xyz} \geq 1$ then $xyz = u^3$ and using inequality (**1**) we obtain $\dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z}+\dfrac{1}{1+u} \geq \dfrac{2}{1+\sqrt{xy}}+\dfrac{2}{1+\sqrt{zu}}$. Since $\sqrt{xy}, \sqrt{zu} \geq 1$ then applying inequality (**1**) we get

$$\frac{1}{1+\sqrt{xy}}+\frac{1}{1+\sqrt{zu}} \geq \frac{2}{1+\sqrt{\sqrt{xy}\cdot\sqrt{zu}}} = \frac{2}{1+\sqrt[4]{xyzu}} = \frac{2}{1+\sqrt[4]{u^3\cdot u}} = \frac{2}{1+u}.$$

Hence, $\dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z}+\dfrac{1}{1+u} \geq \dfrac{4}{1+u} \Leftrightarrow \dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z} \geq \dfrac{3}{1+u} \Leftrightarrow$

(**2**) $\dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z} \geq \dfrac{3}{1+\sqrt[3]{xyz}}$.

Also, since $x,y,z \geq 1$ implies $\sqrt{xy}, \sqrt{yz}, \sqrt{zx} \geq 1$ then by replacing (x,y,z) in inequality (**2**) with $\left(\sqrt{xy}, \sqrt{yz}, \sqrt{zx}\right)$ we obtain

$$\frac{1}{1+\sqrt{xy}}+\frac{1}{1+\sqrt{yz}}+\frac{1}{1+\sqrt{zx}} \geq \frac{3}{1+\sqrt[3]{\sqrt{xy}\cdot\sqrt{yz}\cdot\sqrt{zx}}} = \frac{3}{1+\sqrt[3]{xyz}}$$